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Reliability of Uncertain Flexible Laminated Skewed Plates Under Random Compressions

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A geometrically nonlinear stochastic thin-plate finite element is formulated to study the reliability of fiber-reinforced laminated plates with structural uncertainties under random in-plane loads. The structural uncertainties considered include modulus of elasticity, Poisson's ratio, thickness, fiber orientation of individual lamina, and geometric imperfection of the plate. The deterministic formulation for a 48-degree-of-freedom (DOF) skewed quadrilateral laminated thin-plate element is based on tensorial mathematics and classical laminate theory. The stochastic element formulation is accomplished by including the effect of structural uncertainties and random in-plane loads. The stochastic solution procedure is developed base on the mean-centered second-order perturbation technique together with the Newton-Raphson iterative method. The Hill-Tsai failure criterion for unidirectional lamina and a displacement criterion for the plate are used to set up the limit state functions. First-order reliability methods are adopted to derive the safety index. To demonstrate the applicability of the present developments, a series of fracture and postbuckling analyses of thin plates with structural uncertainties under random in-plane loads are performed. The results quantify the effects of these uncertain parameters on the reliability of the example laminated plates under in-plane loads. The results also show the capability of the current method in analyzing and designing the laminated structures with uncertain parameters subjected to random loads.

I. Introduction

THIN laminated plates are a popular and useful form of structural components with many significant applications in aerospace and other fields of engineering. For the obvious advantages, such as the high strength-to-weight ratio and high stiffness-to-weight ratio, fiber-reinforced laminated composite plates have been increasingly used in thin-wall structures. Because of difficulty frequently encountered in achieving required quality control of the manufacturing process, structural variabilities may occur and are random in nature. These process-induced variabilities may include fiber size, volume fraction of fiber, fiber orientation, thickness of each lamina, lamina strength, and curvature of the plate. Such variabilities will affect the achievable performance of the designed plates and influence their structural reliabilities and serviceabilities.

In recent years, the stochastic finite element method (SFEM) has been widely used to study the effects of material, geometrical, and loading uncertainties on structural behaviors. The method is developed based on the mean-centered second-order perturbation technique and has been proven to be quite effective and accurate for analyzing structures with small property fluctuations. For example, Handa and Anderson[1] solved the linear static response of beams and trusses with uncertain material properties under random loads. Hisada and Nakagiri[2] discussed the stress intensity factor for a crack with uncertain length and the stresses in a plate with fluctuating shape. The expectations and variances of deflection and stress of boron-epoxy laminated plate with random lamina angle orientations and thicknesses under compression were studied by Tani et al.[3] Liaw and Yang[4,5] calculated the reliability of laminated plates and shells with material and geometrical uncertainties using buckling and flutter as failure criteria. Recently, the SFEM has been extended to include the effect of nonlinearities. Liu et al.[6,7] proposed a probabilistic finite element method to analyze the static and dynamic response of elastic-plastic beams and plates with uncertain material properties and uncertain yielding stress subjected to random loads. Hisada[8] developed an analysis method to study the gradients of the respective random variables on the nonlinear response of an elastic-plastic truss structure.

The above-mentioned results demonstrated that the SFEM is useful in predicting, among other results, means and variances of uncertain responses. Such second-moment statistical information can provide insight into the effects of uncertain parameters on structural behavior. However, this information alone is not adequate for reliability analysis, since most engineering structures are designed with a high degree of reliability, which cannot be well described by the first two moments. Calculation of structural reliability can be achieved by first filtering the uncertain structural response calculated by SFEM and the statistical properties of the random variables through the failure criteria assumed and then performing the numerical evaluation of the probability of failure events. The calculation can be difficult to perform if the number of non-normally distributed random variables exceeds two or the failure function becomes nonlinear. Several algorithms have been developed to handle such difficulty. For example, Rackwitz and Fiessler[9] proposed a first-order reliability method (FORM), which calculated the safety index through approximating the nonlinear limit state surface by a tangent hyperplane and transforming the non-normally distributed random variables to equivalent normally distributed ones. Fiessler et al.[10] and Breitung,[11] among others, developed second-order reliability methods (SORMs) through approximating the limit state surface by a quadratic Taylor expansion. Recently, Wu and Wirsching[12] proposed a new algorithm that employed an optimization routine to approximate non-normally distributed random variables by equivalent normally distributed ones. The sensitivity and importance measures of individual stochastic variables in structural reliability have been discussed by, among others, Hohenbichler and Rackwitz[13] and Bjerager and Krenk.[14] They demonstrated that the random variable with an absolutely large alpha-value was considered to be stochastically important compared to the random variables with absolutely small alpha-values.

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The implementation of SFEM together with FORM or SORM for the reliability analysis of complex structures has progressed steadily but mainly in dealing with linear problems. For example, the reliability of a linear uncertain beam-frame structure subjected to random loads was calculated by Hisada and Nakagiri.[15] Kiureghian and Ke[16] studied the reliability of a fixed end beam with stochastic rigidity and a plate with stochastic elasticity. Studies of the reliability of laminated plate with uncertain material parameters were reported by Maekawa et al.[17] and by Tani et al.[18] using strength failure criteria.

The objective of this study is to formulate a geometrically nonlinear stochastic thin-plate finite element to study the reliability of fiber-reinforced laminated thin plates with structural uncertainties under random in-plane loads. The structural uncertainties considered include modulus of elasticity, Poisson's ratio, thickness, fiber orientation of individual lamina, and the geometric imperfection of the plate. The uncertainties are assumed to be space-wise fully correlated throughout the laminae. The deterministic version of the formulation for the 48-DOF skewed quadrilateral laminated thin-plate element was developed by Yang and Wu[19] based on tensorial mathematics and classical laminate theory. The stochastic solution procedure is developed here based on the mean-centered second-order perturbation together with the Newton-Raphson iterative method. The Hill-Tsai failure criterion for the unidirectional lamina and a displacement criterion for the plate are used to set up the limit state function. Rackwitz-Fiesler FORM is adopted to derive the safety index. Five examples are illustrated and discussed to validate the present development. The standard deviations of deflections and layer stresses of a six-ply linear laminated plate with random fiber orientation and the thickness of each layer are first solved and compared with alternative solutions. The reliabilities of a three-layer angle-ply linear laminated constant plane stress plate with random material properties are then calculated to demonstrate the accuracy of the FORM. Finally, three post-buckling examples including an isotropic clamped skewed plate, a simply supported two-layer cross-ply laminated square plate, and a clamped skewed four-layer angle-ply laminated plate with uncertain material properties and geometric imperfections under random in-plane loads are studied. The stochastic importance of each individual random variable is discussed quantitatively through the alpha-value calculated. The results quantify the effects of these uncertain parameters on the reliability of the plates considered. The results also show the capability of the current method in analyzing and designing the laminated structures with uncertain parameters subjected to random loads.

II. Formulation

The formulation of a deterministic 48-DOF skewed quadrilateral thin-shell finite element including the effect of geometrical nonlinearity by Yang and Wu[19] is extended here to include the stochastic effect. The skewed quadrilateral element has 12 DOF at each of the four corner nodes: u, u_ξ, u_η, $u_{\xi\eta}$, v, v_ξ, v_η, $v_{\xi\eta}$, w, w_ξ, w_η, $w_{\xi\eta}$, where u, v, and w are the displacements in the nodal local coordinate ξ and η system.

A. Strain-Displacement Relations

The strain-displacement relations for the imperfect plates are represented in terms of local coordinate components, which are an extended version of the strain-displacement relations given in tensorial form by Mason.[20] The effects of imperfections are included by modifying the tangential strain-displacement relations. These are given as

$$\boldsymbol{\varepsilon}_{\alpha\beta} = \tfrac{1}{2}(\boldsymbol{a}_\alpha \cdot \boldsymbol{u}_{,\beta} + \boldsymbol{a}_\beta \cdot \boldsymbol{u}_{,\alpha} + \boldsymbol{u}_{,\alpha} \cdot \boldsymbol{u}_{,\beta} + \boldsymbol{u}_{,\alpha} \cdot \boldsymbol{s}_{,\beta} + \boldsymbol{u}_{,\beta} \cdot \boldsymbol{s}_{,\alpha}) \tag{1}$$

where $\boldsymbol{a}_\alpha$, $\boldsymbol{u}$, and $\boldsymbol{s}$ are the base vectors of the middle surface, displacement vector, and imperfection vector at a given point on the plate surface with reference to the nodal point local coordinate system, respectively; α and β vary from 1 to 2. In this study, the effects of imperfections on the curvature-displacement relations are ignored.

B. Laminate Constitutive Relations

The laminated anisotropic construction of the plate is assumed to be made up of n layers. Each lamina is assumed to be orthotropic with its principal material axes at an angle with the local coordinate axes. The stress-strain relation for each layer is expressed with reference to the local coordinate system through coordinate transformation. The stress and moment resultants are then related to the middle surface strain ε and the change of curvature κ as[19]

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} \tag{2}$$

where $\{N\}$ is the vector of resultant tangential forces, $\{M\}$ is the vector of resultant moments, and the coefficients in matrices $[A]$, $[B]$, and $[D]$ are given as

$$[A_{ij}, B_{ij}, D_{ij}] = \int_{-t/2}^{t/2} Q_{ij}(1, z, z^2)\,\mathrm{d}z(i, j = 1, 2, 6) \tag{3}$$

with Q_{ij} denoting the plane-stress stiffness for the individual layer and t the total thickness of the plate.

C. Stochastic Finite Element Method

The random field of a single uncertain parameter considered in this study is assumed to be space-wise fully correlated throughout the laminae. The random amplitude ξ_i of the parameter can be expressed as the sum of its mean value $\langle\xi_i\rangle$ and a random variable x_i as

$$\xi_i = \langle\xi_i\rangle + x_i \tag{4}$$

The mean value of random variable x_i is equal to zero; thus, it has the same standard deviation as ξ_i.[4]

The equilibrium equations for the geometrically nonlinear stochastic finite element model can be written as

$$\boldsymbol{P}(\boldsymbol{u}, \boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}) \tag{5}$$

where $\boldsymbol{P}$ is the internal force vector, $\boldsymbol{F}$ is the external force vector, and $\boldsymbol{x}$ is the random parameter vector, which may include random laminate fiber orientation, modulus of elasticity, Poisson's ratio, thickness, geometric imperfection of the plates, or uncertain random loads, etc. Expanding $\boldsymbol{P}$, $\boldsymbol{F}$, and $\boldsymbol{u}$ based on the Taylor series with respect to $\boldsymbol{x}$ and retaining up to second-order terms yield

$$\boldsymbol{u} = \boldsymbol{u}_0 + \sum_{i=1}^{N} \boldsymbol{u}_{x_i} x_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{u}_{x_i x_j} x_i x_j \tag{6}$$

$$\boldsymbol{F} = \boldsymbol{F}_0 + \sum_{i=1}^{N} \boldsymbol{F}_{x_i} x_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{F}_{x_i x_j} x_i x_j \tag{7}$$

$$\boldsymbol{P} = \boldsymbol{P}_0 + \sum_{i=1}^{N} (\boldsymbol{P}_u \boldsymbol{u}_{x_i} + \boldsymbol{P}_{x_i}) x_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\boldsymbol{P}_{uu} \boldsymbol{u}_{x_i} \boldsymbol{u}_{x_j} + \boldsymbol{P}_{u x_i} \boldsymbol{u}_{x_j} + \boldsymbol{P}_{u x_j} \boldsymbol{u}_{x_i} + \boldsymbol{P}_u \boldsymbol{u}_{x_i x_j} + \boldsymbol{P}_{x_i x_j}) x_i x_j \tag{8}$$

The symbol N denotes the total number of random variables. Substituting Eqs. (6–8) into Eq. (5), the random displacement vector is found as follows:

For zeroth order

$$P_0 = F_0 \tag{9}$$

For first order (coefficient of x_i)

$$P_u u_{x_i} = F_{x_i} - P_{x_i} \tag{10}$$

For second order (coefficient of $x_i x_j$)

$$P_u u_{x_i x_j} = F_{x_i x_j} - P_{uu} u_{x_i} u_{x_j} - P_{u x_i} u_{x_j} - P_{u x_j} u_{x_i} - P_{x_i x_j} \tag{11}$$

Equation (9) is nonlinear involving u_0 up to the third order. It can be solved by a standard linear incremental approach in combination with iterations to achieve equilibrium at each increment. The detail numerical procedure can be found in Ref. 19. Equations (10) and (11) are linear in u_{x_i} and $u_{x_i x_j}$, respectively. They can be solved by substituting u_0 into P_u in Eqs. (10) and (11) and then moving the inverse of P_u to the right-hand side of the equations.

After the random displacement vector is calculated, the random stresses in the kth layer can be derived as

$$\sigma_{12}^k(u, x) = T(x)^k Q(x)^k (\varepsilon(u, x) + z^k \kappa(u, x)) \tag{12}$$

where σ_{12}^k represents the stresses in the material principal direction, T^k the coordinate transformation matrix, and z^k the vertical distance between the middle surface and the center of the kth layer. The zeroth, first, and second order stress vectors can be found by expanding Eq. (12) using Taylor series with respect to x as shown in Eqs. (6–8).

D. First-Order Reliability Method

The random displacement and random stress vectors obtained in the previous section can be incorporated into the first-order reliability method to derive safety index β and probability of failure P_f. The probability of failure is defined as

$$P_f = P[g(x) < 0] = \int_\Omega f(x)\, dx \tag{13}$$

where $f(x)$ is the joint probability density function of x, Ω represents the failure region, and $g(x)$ is the so-called limit state function or performance function, which separates the design space into failure ($g(x) < 0$) and safe ($g(x) > 0$) regions. Two types of limit state functions are considered in this study. The first one concerns the nodal displacement at a designated point exceeding an allowable value u_{cr},

$$g(x) = u_{cr} - u(x) \tag{14}$$

The second performance function is constructed at the lamina level using random lamina stresses σ_1, σ_2, τ_{12} and random lamina strengths X, Y, S through the Hill-Tsai criterion,

$$g(x) = 1 - \left(\frac{\sigma_1}{X}\right)^2 + \left(\frac{\sigma_1\sigma_2}{X^2}\right) - \left(\frac{\sigma_2}{Y}\right)^2 - \left(\frac{\tau_{12}}{S}\right)^2 \tag{15}$$

To calculate P_f using FORM, the non-normally distributed random vector x is first transformed to N independent, standard normal random variables Z_i as

$$Z_i = \frac{x_i - \mu_{x_i}}{\sigma_{x_i}} \tag{16}$$

where μ_{x_i}, σ_{x_i} are the mean value and standard deviation of x_i, respectively. The idea of deriving P_f using FORM is to find the minimum distance β (safety index) between the origin and the limit state surface $G = [Z_i; g(Z) = 0]$ in the standard normal domain. By assuming that there is only one minimum distance point Z^* on the limit state surface G, the limit state function $g(Z)$ can be expanded by its first-order Taylor series at Z^*,

$$g(Z) = \nabla g(Z^*)^T (Z - Z^*) \tag{17}$$

where $\nabla g(Z^*)$ denotes the gradient of the function $g(Z)$ at Z^*, the safety index β is found by letting

$$Z^* = \beta\alpha \tag{18}$$

with

$$|\alpha| = 1 \tag{19}$$

The α is called the alpha-value vector. The ith component of α can be interpreted as the importance measure for the corresponding random variable x_i. The random variable with absolutely large alpha-value is considered to be stochastically important compared to those with absolutely small alpha-values. The safety index β described in Eqs. (17–19) can be solved by the Rackwitz-Fiessler iterative scheme.[9] The probability of failure P_f is then derived as

$$P_f = \Phi(-\beta) \tag{20}$$

where Φ is the standard normal distribution function.

III. Numerical Examples

To evaluate the present geometrically nonlinear stochastic finite element formulation and solution procedure, to obtain results for quantifying the effects of the various uncertain parameters, and to gain physical insight into the subject problem, we performed a series of fracture and postbuckling analyses and related reliability studies for thin plates with structural uncertainties subjected to random compressions.

A. Linear Symmetric Laminated Plate with Uncertain Lamina Angle Orientation and Thickness Subjected to Uniaxial In-Plane Load

A six-ply symmetric boron-epoxy laminated plate with lay-up $[-30/30/90]_s$ as shown in Fig. 1 with uncertain lamina angle orientation and thickness subjected to uniaxial in-plane load was first studied to test the accuracy of the present SFEM formulation. The material properties of the lamina were assumed as deterministic with $E_1 = 206.9$ GPa (30×10^6 psi), $E_2 = 20.69$ GPa (3×10^6 psi), $G_{12} = 6.89$ GPa (10^6 psi), $\nu_{12} = 0.3$. The six lamina angle orientations and thicknesses were assumed as independent normally distributed random variables. The expected values of the lamina angle orientations were assumed as $[-30/30/90]_s$, and the standard deviations (STDs) were assumed as 2 deg for each lamina. The expected values and standard deviations of each lamina thickness h were assumed as 0.1 mm (3.94×10^{-3} in.) and 0.001 mm (3.94×10^{-5} in.), respectively.

Sixteen elements were used to model the whole plate. Figure 1 shows the results of the standard deviations of deflections along edges A-B and C-D of the laminated plate. The results obtained by Tani et al.[3] using 128 triangular 15-DOF elements are also plotted in Fig. 1 for comparison. Good agreement is observed. It is seen that the influence of the uncertain lamina angle orientations and thicknesses on the standard deviations of deflection distributes as a nearly symmetrical parabola with the maximum at the midspan of the plate. Figure 2 shows the results of coefficient of variation (COV) of stresses between second (30 deg) and third (90 deg) layers along line I-J of the plate. The results obtained by Tani, et al.[3] are also shown. Except a small discrepancy near point J, fairly good agreement is seen. All three curves exhibit the highest values at the loading point I. The curve for σ_y is obviously higher than the other two along the entire length IJ.

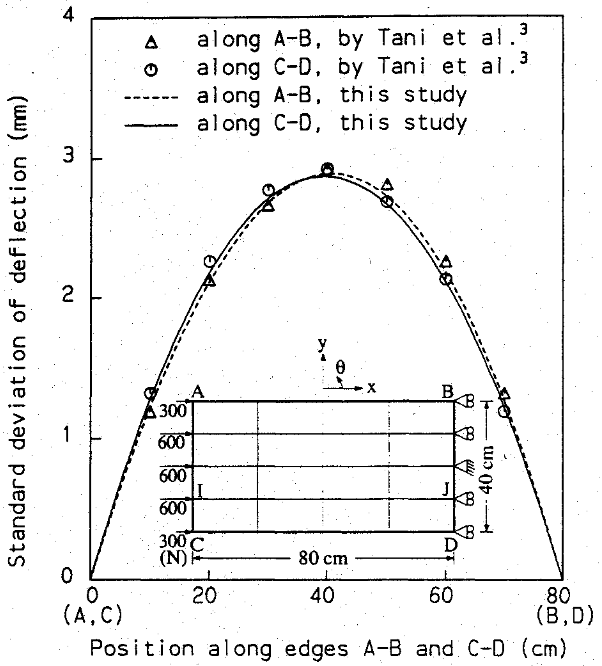

Fig. 1 Standard deviations of deflection along edges *A-B* and *C-D* of a laminated plate $[-30/30/90]_s$ with random layer orientations (STD(θ) = 2 deg) and thicknesses (COV(h) = 1%).

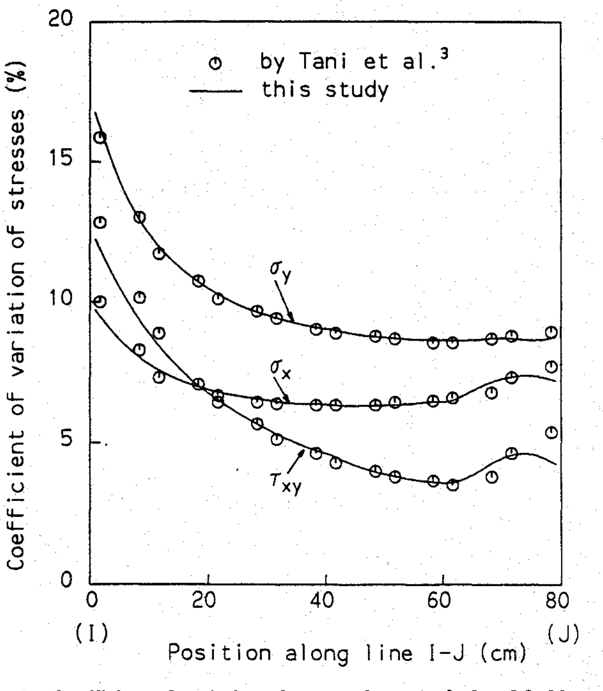

Fig. 2 Coefficient of variation of stresses between 2nd and 3rd layers along line *I-J* of laminated plate $[-30/30/90]_s$ with random layer orientations and thicknesses.

B. Linear Three-Layer Angle-Ply Constant Plane Stress Laminated Plate with Uncertain Material Properties Subjected to Uncertain Multiaxial Loads

The reliability of three-layer angle-ply constant plane stress glass-epoxy laminated plate $[15/-15/15]$ with uncertain material properties subjected to uncertain multiaxial loads was studied to verify the accuracy of the Rackwitz-Fiessler first-order reliability method used. The state of stresses within a single lamina was assumed to be constant and can be derived directly from Eqs. (2), (3), and (12). The expected values of the lamina material properties were assumed as $E_1 = 29.96$ GPa (4.34×10^6 psi), $E_2 = 8.45$ GPa (1.23×10^6 psi), $G_{12} = 4.0$ GPa (0.58×10^6 psi), $\nu_{12} = 0.32$. The expected value of the lamina thickness h was assumed as 1.27×10^{-4} m (0.005 in.). The COVs for E_1, G_{12}, h, and loadings σ_x, σ_y were all assumed to be 5%. The standard deviations of the lamina angle orientations θ were assumed to be 2 deg. The random variables E_1, G_{12}, h, σ_x, σ_y, and θ were all assumed to be normally distributed. It was further assumed that E_1, G_{12}, h, and θ were fully correlated among the three laminae. The failure criterion used in this example is based on that suggested by Yamada and Sun,[21] which is a degenerated form of the Hill-Tsai criterion [Eq. (15)] with $\sigma_2 = 0$. The lamina strengths X and S were assumed to be Weibull distributed with two-parameter distribution functions of the form

$$F(X) = 1 - \exp\left[-\left(\frac{X}{m}\right)^a\right] \tag{21}$$

where a and m are the shape and scale parameters, respectively. The shape and scale parameters for X and S were assumed to be $a_X = 20.5$, $a_S = 17.65$, and $m_X = 0.876$ GPa (1.27×10^5 psi), $m_S = 0.054$ GPa (7.83×10^3 psi). The mean values of X and S were 0.848 GPa (1.23×10^5 psi) and 0.051 GPa (7.45×10^3 psi), respectively.

Seven different cases were studied for two loadings, one in $x(\sigma_x)$ and one in $y(\sigma_y)$ directions, respectively. For the first five cases, one of the five variables $[E_1, G_{12}, h, \theta$, or $\sigma_x(\sigma_y)]$ was assumed random, and the rest of the four variables and strengths X, S were kept as deterministic. For the sixth case, the strengths X and S were assumed random with zero correlation between them, and the five variables were kept as deterministic. For the last case, all five variables and strengths X, S were assumed random with zero correlation among them.

The results of the safety index β for the seven cases subjected to loadings in x and y directions were plotted in Figs. 3 and 4, respectively. Alternative results using the Monte Carlo simulation method with 50,000 samples were also shown. Good agreement between the two results is observed. It is seen that, for the seven cases studied, the case with zero correlation among the seven random variables produces the lowest reliability curve for loading in both x and y directions. Furthermore, for loading in the x direction (Fig. 3), the curve with only one random variable G_{12} shows the highest reliability. Within the range of loading $\langle\sigma_x\rangle$ between 138 MPa (2.0×10^4 psi) and 241 MPa (3.5×10^4 psi), the reliability curve with two random variables X and S is the closest to that of the one with zero correlation among all uncertainties. For the loading in y direction (Fig. 4), the curve with only one random variable E_1 shows the highest reliability. The curve with random variable θ is closest to that of the one with zero correlation among all uncertainties.

Figures 5 and 6 show the alpha-value of the seven random variables considered in the zero correlation case for loadings in the x and y directions, respectively. An attempt was made to arrange the random parameters in the order of their overall significance, i.e., S, θ, h, σ_x, E_1, G_{12}, X in x direction and θ, S, h, σ_y, G_{12}, E_1, X in y direction, respectively. For the presently assumed mean values, COVs, and STDs, it is seen in Fig. 5 that, as the loading in x direction $\langle\sigma_x\rangle$ increases from 138 MPa to 310 MPa, the alpha-value decreases from 0.91 to 0.40 for variable S and increases from 0.35 to 0.74 for variable θ. The alpha-value increases from 0.16 to 0.34 for h, from 0.14 to 0.34 for σ_x, and from 0.11 to 0.24 for E_1. The alpha-values for G_{12} and X remain insignificantly low. It is seen in Fig. 6 that, as the loading in y direction $\langle\sigma_y\rangle$ increases from 7 MPa to 21 MPa, the alpha-value increases from 0.78 to 0.86 for variable θ and decreases from 0.61 to 0.34 for variable S. The alpha-value increases slightly from 0.11 to 0.26 for var-

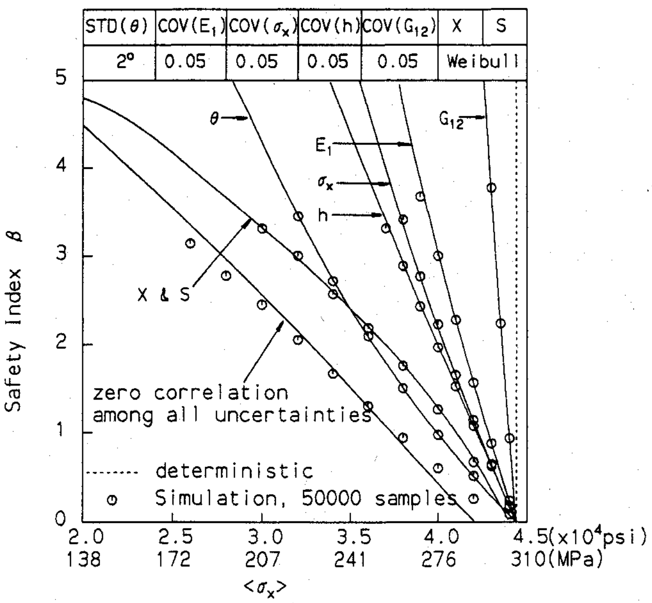

Fig. 3 Safety index β of an angle-ply laminated plate [15/−15/15] with random material properties (θ, E_1, h, G_{12}, X, S) subjected to random loading (σ_x) in x direction.

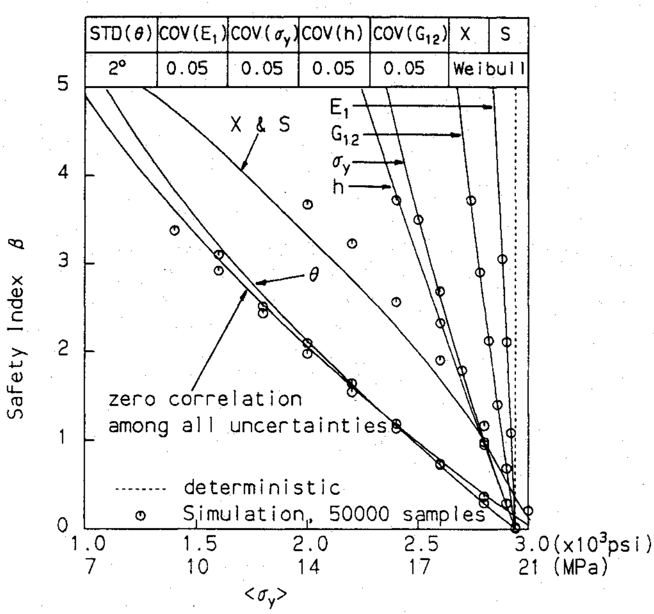

Fig. 4 Safety index β of an angle-ply laminated plate [15/-15/15] with random material properties (θ, E_1, h, G_{12}, X, S) subjected to random loading (σ_y) in y direction.

iable h and from 0.10 to 0.26 for variable σ_y. The alpha-values for G_{12}, E_1, and X remain insignificantly low. For the present two specific cases, the random variables S and θ are stochastically more critical than other variables in both loading in x and y directions. Of course, different assumption of the random variable values may alter this observation.

Figure 7 shows the contours of the safety index of the plate with zero correlation among the random variables E_1, G_{12}, h, θ, σ_x, σ_y, and X, S. The mean loadings were increased from −552 MPa (-8×10^4 psi) to 552 MPa (8×10^4 psi) in x direction and −41 MPa (-6×10^3 psi) to 41 MPa (6×10^3 psi) in y direction. The figure shows that the range enclosed by the two allowable loads shrinks as β value increases (the probability of failure decreases). Construction of such contours may be useful to the design of more reliable structures.

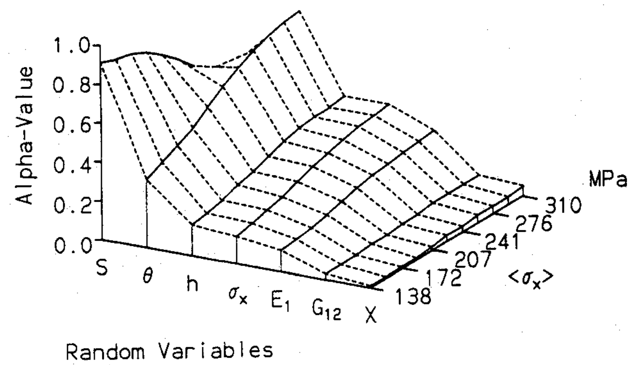

Fig. 5 Alpha-value of random variables considered in Fig. 3 with zero correlation among all uncertainties.

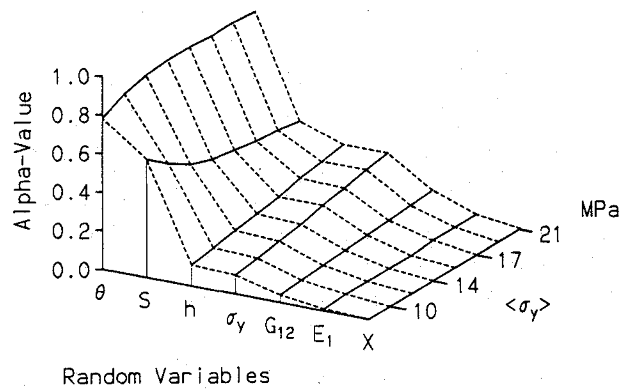

Fig. 6 Alpha-value of random variables considered in Fig. 4 with zero correlation among all uncertainties.

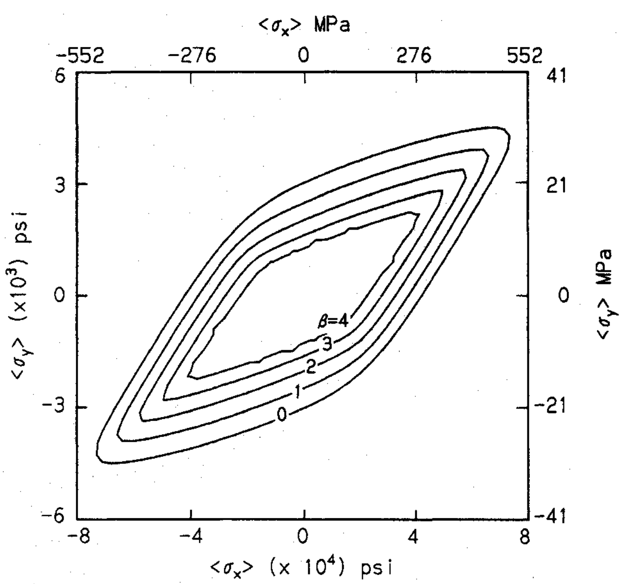

Fig. 7 Contours of safety index of the angle-ply laminated plate [15/−15/15] with random material properties subjected to multiaxial random loading.

C. Postbuckling of an Isotropic Clamped Skewed Plate with Uncertain Material Properties and Geometric Imperfection Subjected to Random In-Plane Loads

The example was to study the postbuckling behavior for an isotropic clamped skewed plate. The plate was assumed to be skewed with angle 15 deg. Due to the skewness of the local coordinate system, the in-plane load in x direction was divided into components in both ξ and η directions. During the postbuckling movement, all four edges were assumed free from in-plane stresses. The length L of the plate was assumed to be deterministic with value 0.254 m (10.0 in.). The mean value $\langle\cdot\rangle$ and coefficient of variation COV($\cdot$) of the material properties of the plate were assumed to be normally distributed with values $\langle E\rangle$ = 206.9 GPa (30 × 10^6 psi), COV(E) = 0.05; $\langle\nu\rangle$ = 0.3, COV(ν) = 0.05; $\langle h\rangle$ = 2.54 × 10^{-3} m (0.1 in.), COV(h) = 0.025. The shape of the geometric imperfection was assumed with the following form:

$$w_i(x, y) = |\mu|\frac{\langle h\rangle}{4}\left(1 - \cos\frac{2\pi\xi}{L}\right)\left(1 - \cos\frac{2\pi\eta}{L}\right) \tag{22}$$

where μ is the nondimensionalized amplitude of the imperfection with $\langle\mu\rangle$ = 0.0, STD = 0.05$\langle h\rangle$. The loading N_x was assumed to be normally distributed with COV(N_x) = 0.05.

The plate with deterministic properties subjected to deterministic in-plane load was first calculated to test the accuracy of the present geometrically nonlinear finite element formulation. Sixteen elements were used to model the whole plate. The results for nondimensionalized total center deflection are plotted in Fig. 8 against nondimensionalized in-plane load for two cases, μ = 0.0 and 0.1. Alternative solution for μ = 0.0 calculated by Prabhu and Durvasula[22] using the Galerkin method and the Newton-Raphson procedure is also plotted. Good agreement is seen between the two solutions.

The reliability of the plate with random properties and imperfection subjected to uncertain in-plane loads was calculated based on the displacement criterion (Eq. 14). The critical value of the displacement u_{cr} was assumed to be at the center with magnitude 2.0$\langle h\rangle$. Six different cases were considered. For the first five cases, one of the five random variables (E, h, N_x, μ or ν) was assumed random while the other four were assumed deterministic. For the last case, all five random variables were assumed random with zero correlation among them. The results of the safety index β for these six cases were plotted in Fig. 9. It is seen that the curve with zero correlation among the five variables shows the lowest reliability, whereas the curve with random variable ν gives the highest reliability. Figure 10 shows the alpha-value of the five random variables considered in the zero correlation case. For the presently assumed mean values, COVs, and STDs, it is seen that, as the nondimensional compressive loading $\langle N_x\rangle L^2/\pi^2 D$ increases from 14.5 to 20.0, the alpha-values for random variables E, h, and N_x are relatively the highest, in the neighborhood of 0.50 to 0.60. The alpha-values are considerably lower for μ and even more so for ν. Thus, for the presently assumed parameter values, the random variables E, h, and N_x are found to be stochastically more critical than the variables μ and ν.

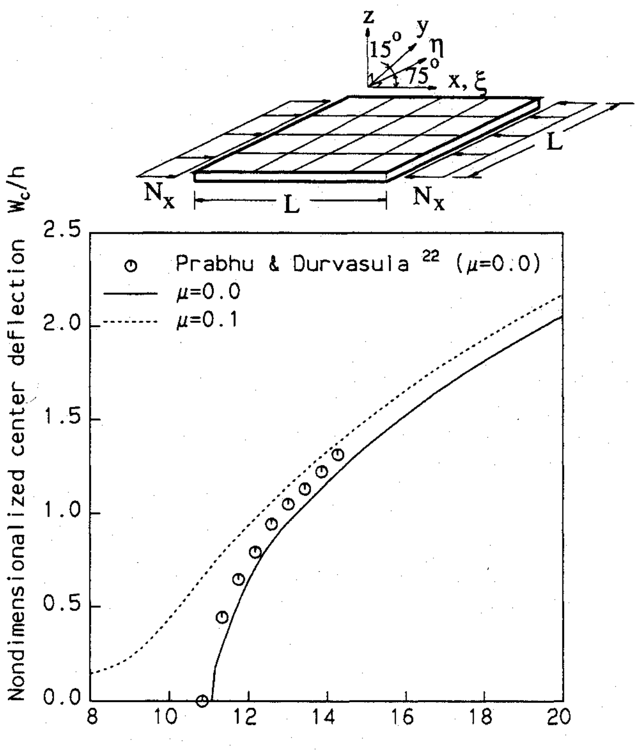

Fig. 8 Postbuckling of an imperfect isotropic clamped skewed deterministic plate with all four edges free from in-plane stresses (4 × 4 element mesh, whole plate).

D. Postbuckling of a Two-Layer Cross-Ply [0.90] Laminated Simply Supported Square Plate with Uncertain Material Properties and Uncertain Geometric Imperfection Subjected to Random In-Plane Load

The example was to study the postbuckling behavior of a two-layer cross-ply [0/90] graphite-epoxy laminated simply supported square plate. During the postbuckling movement, all four edges were assumed as free from in-plane stresses. The length L of the plate was assumed to be deterministic with value 0.254 m (10 in.). The lamina properties were assumed as $\langle E_1\rangle$ = 206.85 GPa (30 × 10^6 psi), COV(E_1) = 0.05; $\langle E_2\rangle$ = 5.17 GPa (0.75 × 10^6 psi), COV(E_2) = 0.0; $\langle\nu_{12}\rangle$ = 0.25, COV(ν_{12}) = 0.0; $\langle G_{12}\rangle$ = 2.586 GPa (0.375 × 10^6 psi), COV(G_{12}) = 0.0; $\langle h\rangle$ = 1.27 × 10^{-4} m (0.005 in.),

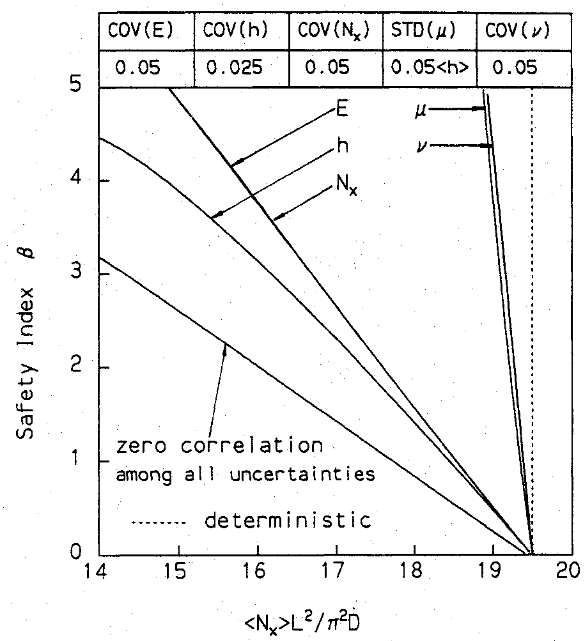

Fig. 9 Safety index β of an isotropic clamped skewed plate with random material properties subjected to uniaxial random loading.

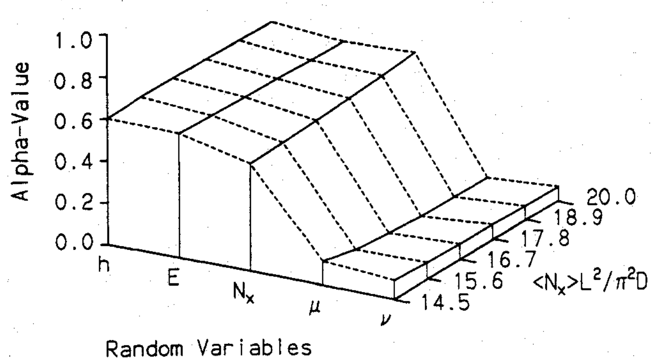

Fig. 10 Alpha-value of random variables considered in Fig. 9 with zero correlation among all uncertainties.

COV(h) = 0.05. These properties were assumed to be normally distributed and fully correlated among the two layers. The mean values of the two lamina angle orientations were assumed as 0 deg and 90 deg, respectively. The standard deviations were assumed to be 2 deg. The shape of the geometric imperfection was also assumed with the following form:

$$w_i(x, y) = |\mu|\langle 2h\rangle \sin \frac{\pi x}{L} \sin \frac{\pi y}{L} \tag{23}$$

where the nondimensionalized imperfect amplitude μ was assumed normally distributed with mean = 0.0, STD = 0.1$\langle h\rangle$. The loading N_x was assumed to be normally distributed with COV(N_x) = 0.05.

The postbuckling response of the laminated plate with deterministic properties subjected to deterministic load was first calculated using four elements for a quarter of the plate. The results for the total nondimensionalized center deflection with respect to nondimensionalized load were plotted in Fig. 11 with μ = 0.0, 0.1, and 0.25, respectively. An alternative solution calculated by Prabhakara[23] with μ = 0.0 is also shown in the figure. Good agreement is observed between the two solutions.

The reliability of the plate with random properties and imperfection subjected to uncertain in-plane loads was calculated based on the same failure criterion as in example 3 with u_{cr} equal to twice the total thickness of the plate. Six different cases were considered. For the first five cases, one of the five variables E_1, h, θ, N_x, or μ was assumed random while the other four were assumed to be deterministic. For the last case, all five variables were assumed random with zero correlation among them. The results of the safety index β for the six cases were plotted in Fig. 12. It is seen that the case with all five variables random yields the lowest reliability curve. Also, when considering the individual effect of only one variable on the reliability of the plate, the random variable h produces the lowest curve. Figure 13 shows the alpha-value of the five random variables considered in the zero correlation case. For the presently assumed mean values, COVs, and STDs, it is seen that, as the nondimensional compressive loading $\langle N_x\rangle L^2/E_2\langle 2h\rangle^3$ increases from 8 to 18, the alpha-value

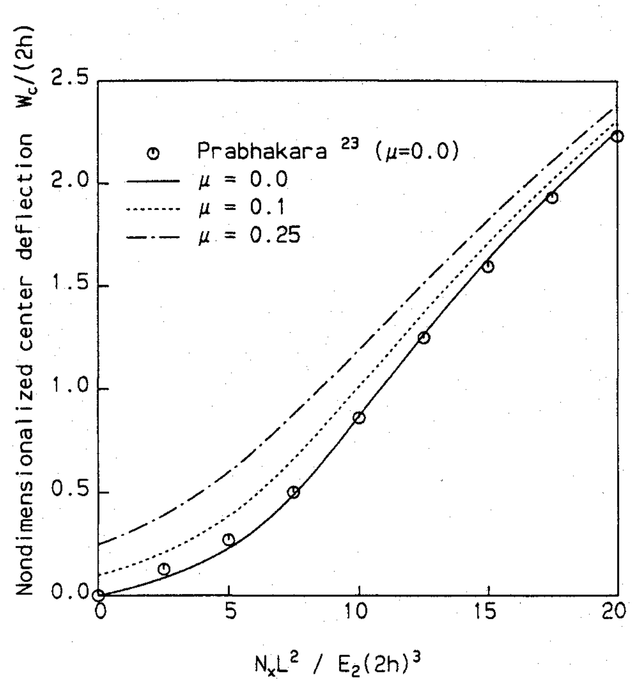

Fig. 11 Postbuckling of a cross-ply laminated imperfect simply supported deterministic square plate [0/90] (2 × 2 element mesh, a quarter plate).

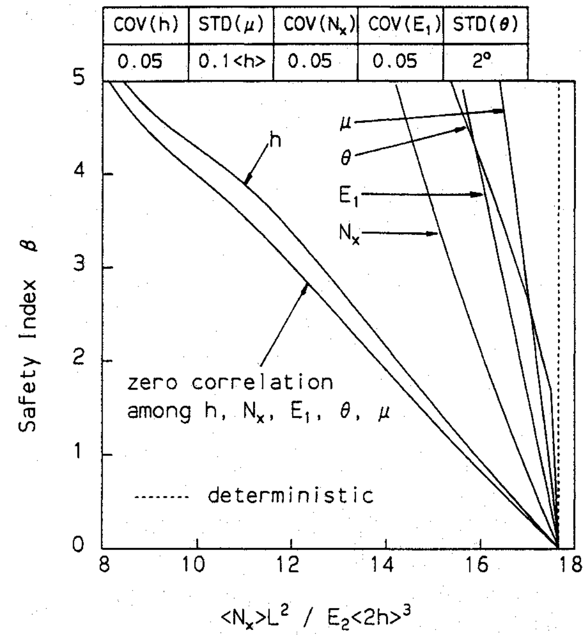

Fig. 12 Safety index β of a cross-ply laminated plate [0/90] with random material properties subjected to uniaxial random loading.

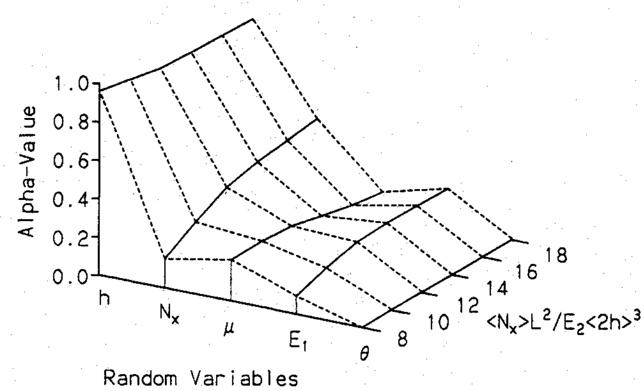

Fig. 13 Alpha-value of random variables considered in Fig. 12 with zero correlation among all uncertainties.

decreases slightly from 0.96 to 0.88 for variable h, decreases from 0.22 to 0.11 for variable μ, increases from 0.15 to 0.42 for variable N_x, and increases from 0.09 to 0.20 for variable E_1. The alpha-value for θ remains insignificantly low. Thus, for the presently assumed parameter values, the random variable h is found to be stochastically more critical than the other four variables E_1, θ, N_x, and μ.

E. Postbuckling of a Four-Layer Angle-Ply $[\pm 45]_2$ Laminated Clamped Skewed Plate with Uncertain Material Properties Subjected to Random In-Plane Load

The last example was to study the postbuckling behavior of a four-layer angle-ply $[\pm 45]_2$ glass-epoxy laminated clamped skewed plate. The skew angle was assumed to be 15 deg. During the postbuckling movement, all four edges were assumed free from in-plane stresses. The length L of the plate was assumed to be deterministic with value 0.254 m (10 in.). The lamina properties were assumed as $\langle E_1\rangle$ = 53.78 GPa (7.8 × 10^6 psi), COV(E_1) = 0.05; $\langle E_2\rangle$ = 18.15 GPa (2.63 × 10^6 psi), COV(E_2) = 0.05; $\langle \nu_{12}\rangle$ = 0.25, COV(ν_{12}) = 0.05; $\langle G_{12}\rangle$ = 8.23 GPa (1.19 × 10^6 psi), COV(G_{12}) = 0.05; $\langle h\rangle$ = 1.27 × 10^{-4} m (0.005 in.), COV(h) = 0.02. The mean values of the four lamina angle orientations were assumed as 45, −45, 45, and −45 deg, respectively. The standard deviations were assumed to be 2 deg. These properties were assumed to be normally distributed and fully correlated among the four layers. The shape of the geometric imperfection was assumed with the following form:

$$w_i(x, y) = |\mu|\langle h\rangle\left(1 - \cos\frac{2\pi\xi}{L}\right)\left(1 - \cos\frac{2\pi\eta}{L}\right) \quad (24)$$

where nondimensionalized imperfection amplitude μ was assumed normally distributed with $\langle\mu\rangle = 0.0$, STD $= 0.4\langle h\rangle$. The loading N_x was assumed to be normally distributed with $\mathrm{COV}(N_x) = 0.02$.

The linear buckling load and postbuckling response of the laminated plate with deterministic properties subjected to deterministic load was first calculated using 16 elements for the whole plate. The results for the total nondimensionalized center deflection are plotted in Fig. 14 against nondimensionalized load. For linear buckling load, the alternative result calculated by Kamal and Durvasula[24] using the series expansion method is also shown in the figure. Good agreement is observed between the two values.

The reliability of the plate with random properties subjected to uncertain in-plane loads was calculated based on the same failure criterion as in example 3 with u_{cr} equal to twice the total thickness of the plate. Nine different cases were considered. For the first eight cases, one of the eight variables E_1, E_2, G_{12}, ν_{12}, h, N_x, μ, and θ was assumed random while the other seven were assumed to be deterministic. For the last case, all eight variables were assumed random with zero correlation among them. The results of the safety index β for the nine cases were plotted in Fig. 15. It is seen that the case with all eight variables random yields the lowest reliability curve. Also, when considering the individual effect of only one variable on the reliability of the plate, the random variable h produces the lowest curve. Figure 16 shows the alpha-value of the eight random variables considered in the zero correlation case. For the presently assumed mean values, COVs, and STDs, it is seen that, as the nondimensional compressive

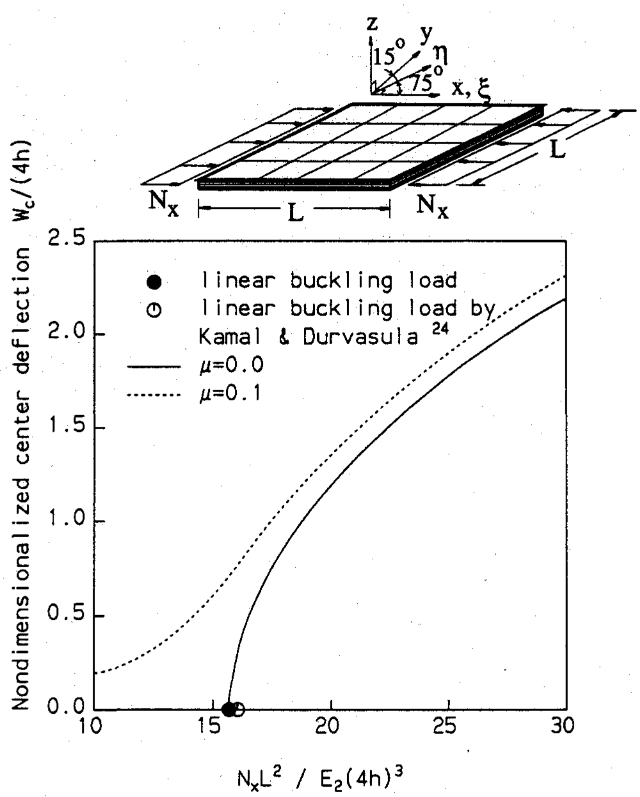

Fig. 14 Postbuckling of an imperfect laminated clamped skewed deterministic plate $[\pm 45]_2$ with all four edges free from in-plane stresses (4 × 4 element mesh, whole plate).

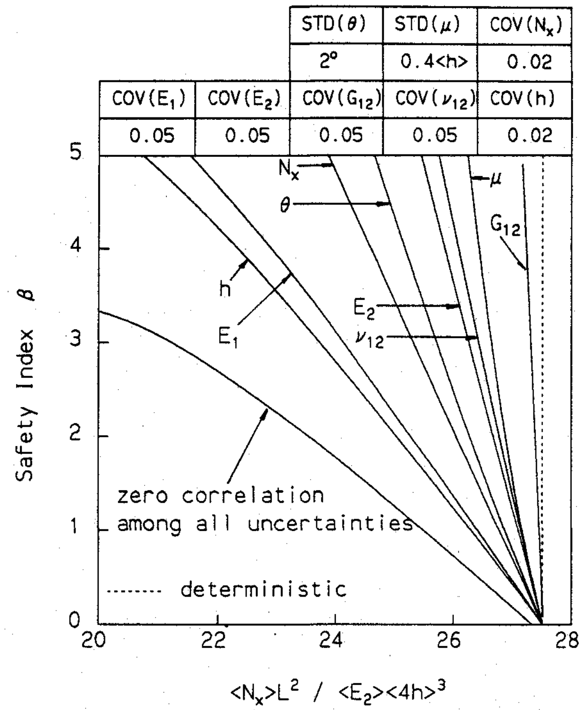

Fig. 15 Safety index β of a laminated clamped skewed plate $[\pm 45]_2$ with random material properties subjected to uniaxial random loading.

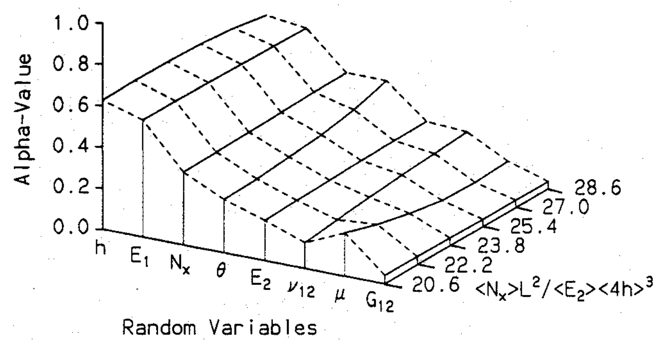

Fig. 16 Alpha-value of random variables considered in Fig. 15 with zero correlation among all uncertainties.

loading $\langle N_x\rangle L^2/\langle E_2\rangle\langle 4h\rangle^3$ increases from 20.6 to 28.6, the alpha-value varies slightly between 0.57 to 0.62 for variable h, between 0.57 and 0.55 for variable E_1, between 0.35 and 0.37 for variable N_x, and between 0.19 and 0.20 for variable E_2; increases slightly from 0.26 to 0.37 for variable θ and from 0.12 to 0.20 for variable ν_{12}; and decreases from 0.21 to 0.09 for variable μ. The alpha-value for G_{12} remains insignificantly low. Thus, for the presently assumed parameter values, the random variables h and E_1 are found to be stochastically more critical than the other six variables E_2, G_{12}, ν_{12}, N_x, μ, θ.

IV. Concluding Remarks

A 48-DOF geometrically nonlinear stochastic thin-plate finite element has been formulated to study the reliability of fiber-reinforced laminated plate with uncertain modulus of elasticity, Poisson's ratio, thickness, fiber orientation of individual lamina, and geometric imperfection subjected to random in-plane loads. The stochastic solution procedure has been developed based on the mean-centered second-order perturbation technique together with the Newton-Raphson iterative method. The Hill-Tsai failure criterion for unidirectional lamina and a displacement criterion for the plate have been used to set up the limit state function. The Rackwitz-

Fiessler first-order reliability method has been adopted to derive the safety index.

The present results have been compared with alternative solutions in the case of random linear analysis (Figs. 1 and 2) and deterministic geometric nonlinear analysis (Figs. 8 and 11). The Monte Carlo simulation method has also been used to verify the accuracy of the FORM on the fracture of a linear constant plane stress laminated plate with material uncertainties (Figs. 3 and 4). In this study, a range of values for each random parameter has been assumed and the results demonstrate the individual effect of these parameters on the reliability of the cases studied. The calculation of alpha-values further quantifies the stochastic importance of these parameters. Such a numerical prediction method and quantified information may be helpful in the design of more reliable laminated composite plate structures.

Acknowledgments

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